*J. Fluid Mech.* (2004), *vol.* 503, *pp.* 1–14. © 2004 Cambridge University Press 1
DOI: 10.1017/S0022112003007389 Printed in the United Kingdom

# A simple hydro-elastic model of the dynamics of a vitreous membrane

By R. REPETTO[1], I. GHIGO[2], G. SEMINARA[2] AND C. CIURLO[3]

[1]Dipartimento di Ingegneria delle Strutture, delle Acque e del Terreno, Università dell'Aquila, Monteluco di Roio, 67040, L'Aquila, Italy

[2]Dipartimento di Ingegneria Ambientale, Università di Genova, Via Montallegro 1, 16145, Genova, Italy

[3]Dipartimento di Neuroscienze, Oftalmologia e Genetica, Università di Genova, Pad. 9, Largo Rosanna Benzi 10, 16132 Genova, Italy

(Received 4 December 2002 and in revised form 9 October 2003)

We study the motion of a fluid within a rigid spherical container subject to small-amplitude periodic rotations. The sphere is divided into two equal portions by an impermeable stretched elastic membrane whose boundary is attached to the container wall. The model aims to represent in a simplified fashion the dynamics of a vitreous membrane subject to microsaccadic movements of the human eye, assuming the vitreous to be liquefied. The vitreous is modelled as a Newtonian, incompressible fluid in irrotational motion and the problem is linearized, taking advantage of the hypothesis of small-amplitude eye rotations. Results show that, due to the presence of the fluid, the natural frequencies of oscillation of the membrane decrease significantly with respect to the case of a free membrane. Moreover, oscillations of a stretched membrane are found to be resonantly excited by rotations of the sphere with frequencies which are typical of microsaccadic eye movements. This study suggests the possibility that oscillations of vitreous membranes may induce the development of large tensile stresses capable of producing a retinal detachment. Such a conclusion will have to be further substantiated by more refined analyses accounting for further effects, such as nonlinearity and the possible viscoelastic behaviour of the vitreous located on one or both sides of the membrane.

## 1. Introduction

The problem investigated in the present work is based on a somewhat idealized model of the dynamics of vitreous membranes. Though the physiological relevance of the present model may require further substantiation in future developments, the subject of vitreous hydrodynamics which motivated the present work provides a useful introduction.

The vitreous body is a transparent gel-like structure located in the posterior chamber of the eye. In the normal eye of an adult, the vitreous occupies a volume of approximately 4 ml and it constitutes from 3/5 to 2/3 of the total volume and weight of the eye globe. The density of the vitreous is equal to $1005.3\,\mathrm{Kg\,m^{-3}}$ (Tolentino, Schepens & Freeman 1976). Its major constituents are water, soluble proteins, hyaluronic acid, collagen and a small population of cells, designated as hyalocytes (Klintworth & Scroggs 1997). From the mechanical point of view the human vitreous

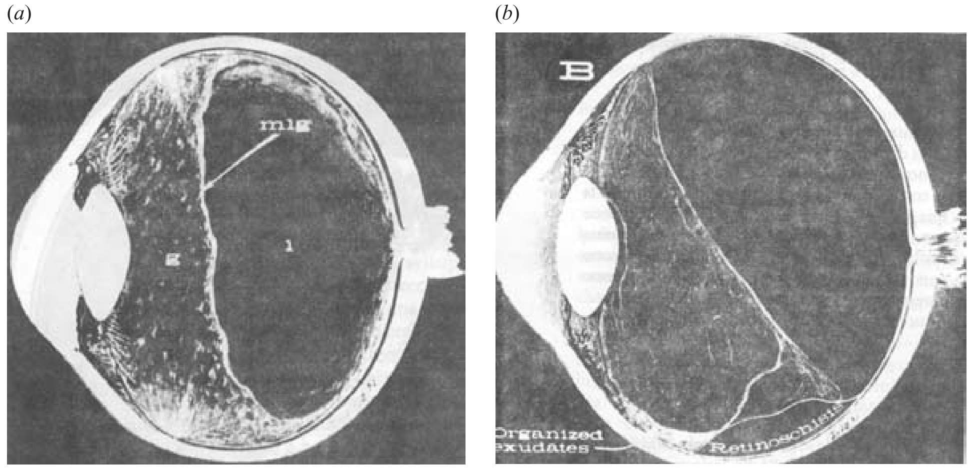


Figure 1. (*a*) Vitreous membrane separating liquefied vitreous from gel; (*b*) stretched vitreous membrane which has caused a retinoschisis, i.e. a break of the retina (from Tolentino *et al.* 1976, with permission from Elsevier).

is a viscoelastic material, whose characteristics depend on age and vary within the vitreous cavity (Lee, Litt & Buchsbaum 1992). The gelatinous consistency of the vitreous is due to a framework of numerous, randomly oriented, collagen fibrils. With advancing age the vitreous often loses its consistency and undergoes a liquefaction process.

Particular intraocular processes, such as inflammatory states and haemorrhages, may lead to the formation of membranes within the vitreous, which split the vitreous body into separate regions (see figure 1*a*, *b*). Vitreous membranes can develop under diverse conditions, indicating that multiple factors may be involved in their pathogenesis. Membranes display highly variable thickness and size and can be loose or stressed. It is customary to classify membranes according to the following criteria: (i) membranes surrounded by gel, (ii) membranes surrounded by liquefied vitreous and (iii) membranes separating gel from liquefied vitreous (Tolentino *et al.* 1976). The formation of vitreous membranes does not necessarily represent a dangerous medical condition; indeed membranes may form in the vitreous cavity without causing damage to the retina or sight reduction. However, as described by Tolentino *et al.* (1976), membranes may produce retinal lesions of different nature if they are attached to the retina and exert stresses on it. Such stresses may arise from a progressive contraction of the membrane (static stresses) or may be generated by eye movements (dynamic stresses). Though the latter classification of stresses induced by membranes on the retina is only based on clinical observations, it is however the basis of several therapies. Therefore, founding clinical practice on a better understanding of vitreous dynamics in the presence of vitreous membranes is desirable.

The hydrodynamics of the normal vitreous during ocular movements has been the subject of recent investigations. David *et al.* (1998) have studied the motion of vitreous humour of the human eye during saccadic movements. The aim of their work was to estimate the shear stress acting on the retina during saccadic movements in order to ascertain the existence of a correlation between retinal detachment and occurrence of high shear stress at the retina. The authors modelled the eye as a rigid sphere subject to sinusoidal oscillations, while vitreous humour was modelled either as a Newtonian fluid or as a viscoelastic material, whose constitutive behaviour was interpreted through a Maxwell–Voigt model. Their results show that, as the so-called Womersley number increases, the flow driven by eye rotations is confined within a Stokes-type layer of increasingly small thickness. This is a feature arising from the almost purely diffusive nature of the flow field induced by the eye motion in the perfectly symmetric geometry considered by David *et al.* The main result of the paper is the suggestion that the maximum shear stress increases linearly with eye radius and with the 3/2 power of the angular frequency of the eye rotations. The larger frequency of retinal detachment in myopic eyes, which are known to be characterized by sizes larger than normal eyes, may thus be related to an increased shear stress. A further interpretation of the higher risk of retinal detachment in myopic eyes had already been given by David *et al.* (1997) who studied the time-dependent shear force on the eye wall shell during saccadic movements. They found that, taking into account the reduction of the eye wall thickness, the increased eye radius and increased muscular force that characterize myopic eyes, the shear force in the eye wall may be up to seven times larger than in normal eyes; this corresponds to a significantly higher risk of retinal detachment.

It is finally worth citing a recent work of González & Fitt (2003) regarding some aspects of the mathematical modelling of human eyes. They have studied the mathematics of tonometry, i.e. a technique which is often employed in the measurement of intraocular pressure. The head of the tonometer is used to compress the frontal part of the cornea: from the force required to flatten the cornea the intraocular pressure is computed on the basis of an empirical principle known as the Imbert–Fick principle, which was established over a century ago. González & Fitt have assumed the eye to be a linearly elastic hollow sphere and have determined a relationship between the flattening force and the internal pressure, showing the Imbert–Fick principle to be valid when the intraocular pressure is not very much higher than its physiological value. The mathematical model developed for the tonometry problem has also been used by González & Fitt to investigate problems related to scleral buckling, a surgical technique often employed to treat retinal detachment. The paper ends with a stimulating account of some open problems concerning the mechanics of the human eye that will deserve attention from researchers in the future.

In the present paper we formulate an idealized hydro-elastic model in order to obtain some insight into the question of whether a vitreous membrane surrounded by liquefied vitreous may respond with large-amplitude oscillations to eye rotations, thus inducing abnormally high tractions on the retina. In particular we focus our attention on high-frequency, small-amplitude eye rotations, which Ashe *et al.* (1991) define “microsaccadic flutter”. Such movements, which are characterized by amplitudes lower than $1^\circ$ and frequencies ranging between 15 and 30 Hz, can barely be observed clinically without special equipment. However, they have the extremely important role of permitting a scan of the fine detail of the image projected onto the fovea, the central region of the retina where the larger concentration of light receptors is found.

The assumption of small-amplitude eye rotations allows us to tackle the problem through a linearized mathematical model. Moreover, we assume the fluid to be Newtonian and we consider a perfectly spherical eye globe separated into two equal regions by an impermeable, stretched, elastic vitreous membrane. The above assumptions are fairly restrictive as the vitreous may not be liquefied on both sides of the membrane and the membrane is not necessarily pre-stressed. Though under admittedly idealized conditions, the paper shows that the natural frequencies of vitreous membranes are strongly reduced by the presence of the fluid phase, to the

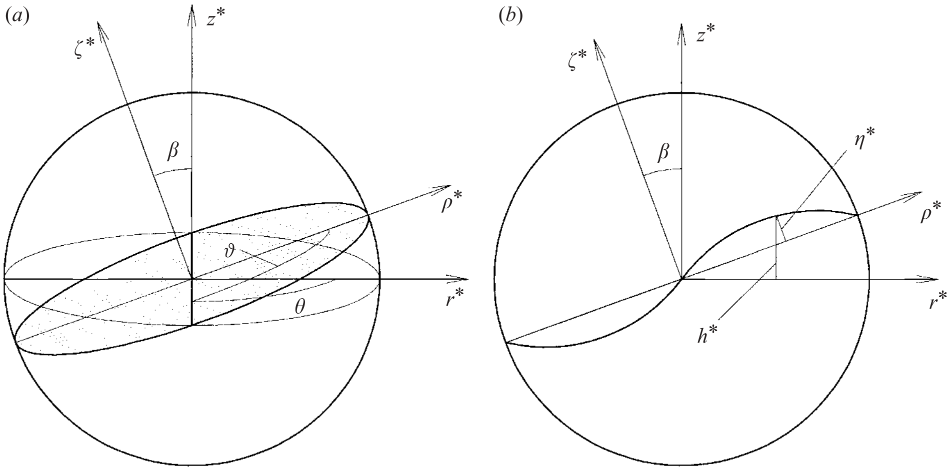


FIGURE 2. (*a*) Sketch of the problem and notation; (*b*) sketch of a globe section and notation.

extent that, for realistic values of the relevant physical parameters, vitreous membranes may resonate at frequencies close to those typical of microsaccadic movements of the eye. Such a finding suggests that the question of whether vitreous membrane oscillations may produce dynamic tensile stresses on the retina such to induce its detachment deserves attention. In order to ascertain whether such a finding has actual clinical relevance, various developments of the present investigation will be needed. In particular, a worthwhile development will be to remove the assumption of linearity in order to be able to evaluate the dynamic stresses developing in the membrane. Moreover, the case of membranes which are not pre-stressed, as well as configurations in which membranes separate liquefied vitreous from gel or are completely surrounded by gel, also deserve attention. Finally, we point out that the presence of vitreous membranes is one among many possible causes of retinal detachment.

The rest of the paper is organized as follows. In § 2 we formulate the mathematical problem describing the dynamics of the vitreous–membrane system. In § 3 we describe the procedure employed to solve the hydrodynamic problem and discuss the results concerning the flow fields. Section 4 is devoted to the solution of the membrane equation; both free vibrations and oscillations forced by eye movements are analysed. Finally, some concluding remarks in § 5 complete the paper.

## 2. Formulation of the problem

We study the three-dimensional flow of a fluid within a spherical container subject to periodic rotations. The interior of the sphere is divided into two equal regions separated by an impermeable, stretched, elastic membrane and the boundary of the membrane is fixed at the container wall. A sketch of the geometry under consideration is given in figure 2(*a*,*b*). Two systems of cylindrical coordinates will be employed: $(r^*, \theta, z^*)$ describes the absolute fluid motion with respect to a fixed coordinate system; and $(\rho^*, \vartheta, \zeta^*)$ will be used to represent the absolute motion in terms of time-dependent coordinates rotating with the sphere. The line identified by $\theta = 0, \pi$ remains coincident with that identified by $\vartheta = 0, \pi$ and, at time $t^*$, the $\zeta^*$-axis is rotated about this line through the angle $\beta(t^*)$ with respect to the $z^*$-axis. An asterisk denotes dimensional variables that will later be made dimensionless. As shown in the sketches presented in figure 2($a,b$) the unperturbed position of the membrane is described by the equation $\zeta^*=0$. Below, we formulate the problem for the fluid contained in the upper region of the container, the problem for the lower fluid being identical.

We treat the liquefied vitreous as an incompressible Newtonian fluid with the same mechanical characteristics (density $\rho_v$ and viscosity) on both sides of the membrane. This hypothesis seems sensible in the case of liquefied vitreous, a situation encountered in practice.

We assume the flow to be irrotational. In fact, we assume that the vorticity generated at the sidewall remains confined within thin Stokes layers adjacent to the boundaries which may be neglected at leading order. This hypothesis is likely to be appropriate everywhere except close to the sharp junction between the membrane and the spherical wall, where a stagnation line of the relative motion occurs. To what extent such an assumption may affect the validity of the analysis can ultimately be ascertained by performing a fully viscous analysis.

The assumption of irrotational flow allows us to introduce a velocity potential, defined as $\nabla\phi^*=\boldsymbol{u}^*$, where $\boldsymbol{u}^*$ is the velocity. The velocity $\boldsymbol{u}_r^*$ relative to a frame rotating with the eye is not irrotational: indeed, one readily finds that $\nabla\times\boldsymbol{u}_r^*=(0,0,2\Omega^*)$, where $\Omega^*=\mathrm{d}\beta/\mathrm{d}t^*$ is the angular velocity of the reference frame. The problem is formulated with respect to a fixed frame, using the cylindrical coordinates $(r^*,\theta,z^*)$ sketched in figure 2. The velocity potential $\phi^*$ being harmonic, we can write

$$\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial\phi^*}{\partial r^*}\right)+\frac{1}{r^{*2}}\frac{\partial^2\phi^*}{\partial\theta^2}+\frac{\partial^2\phi^*}{\partial z^{*2}}=0. \tag{2.1}$$

At the eye wall we impose the requirement of no flux through the boundary:

$$\frac{\partial\phi^*}{\partial r^*}r^*+\frac{\partial\phi^*}{\partial z^*}z^*=0 \qquad (r^{*2}+z^{*2}=R^2), \tag{2.2}$$

with $R$ denoting the eye radius. Furthermore, the kinematic condition at the membrane states that the membrane is a material surface. If the equation of the membrane interface is written in the form $z^*=h^*(r^*,\theta,t^*)$ (see figure 2$b$) the kinematic condition is

$$\frac{\partial h^*}{\partial t^*}+\frac{\partial h^*}{\partial r^*}\frac{\partial\phi^*}{\partial r^*}+\frac{1}{r^{*2}}\frac{\partial h^*}{\partial\theta}\frac{\partial\phi^*}{\partial\theta}-\frac{\partial\phi^*}{\partial z^*}=0 \qquad (z^*=h^*). \tag{2.3}$$

The dynamic pressure $p^*$ enters the problem through Bernoulli's theorem, which, for a fluid in unsteady irrotational motion leads to the following relationship:

$$\frac{p^*}{\rho_v}+\frac{1}{2}\left[\left(\frac{\partial\phi^*}{\partial r^*}\right)^2+\frac{1}{r^{*2}}\left(\frac{\partial\phi^*}{\partial\theta}\right)^2+\left(\frac{\partial\phi^*}{\partial z^*}\right)^2\right]+\frac{\partial\phi^*}{\partial t^*}=0, \tag{2.4}$$

holding throughout the flow field. We recall that $\phi^*$ is defined up to an arbitrary function of time which can be absorbed into $\phi^*$ by suitably redefining the potential function.

At this stage it is convenient to perform the following coordinate transformation:

$$\rho^*=\sqrt{(r^*\cos\theta)^2+(r^*\sin\theta\cos\beta+z^*\sin\beta)^2}, \tag{2.5a}$$

$$\vartheta=\arctan\left(\frac{r^*\sin\theta\cos\beta+z^*\sin\beta}{r^*\cos\theta}\right), \tag{2.5b}$$

$$\zeta^*=z^*\cos\beta-r^*\sin\theta\sin\beta, \tag{2.5c}$$

where, as pointed out above, the coordinate system $(\rho^*, \vartheta, \zeta^*)$, shown in figure 2, rotates with the sphere. The variable $h^*$ can then be written in terms of the new coordinates as

$$h^* = \eta^* \cos\beta + \rho^* \sin\vartheta \sin\beta, \tag{2.6}$$

where $\eta^*$ denotes the displacement of a point of the membrane in the $\zeta^*$-direction. In terms of the new coordinates the differential system (2.1), (2.2) and (2.3) takes a complicated and lengthy form which is not reported here for the sake of brevity. A linearized dimensionless version of these equations will be presented below.

In order to complete the formulation of the problem we need to introduce a further equation describing the motion of the membrane. This equation is coupled with the flow field through the effect of fluid stresses acting on the membrane which are in turn determined by the membrane motion. In the following, viscous stresses associated with the motion of the membrane will be neglected: such an assumption seems reasonable since the normal component of viscous stresses can be expected to be much smaller than the dynamic pressure and the membrane is assumed to be subject to an elastic pre-stress much larger than the viscous tangential stresses induced by fluid motion.

The dynamics of a stretched circular elastic membrane is a classical problem of mathematical physics (see for instance Courant & Hilbert 1937). In the present paper we consider a homogeneous membrane. Moreover, we assume that the stretching force per unit length $T$, i.e. the force per unit length acting on each side of any cut through the membrane, is large enough to justify neglecting the direct effect of angular velocity and acceleration on the membrane. In other words membrane oscillations with respect to its undisturbed position, which rotates with the eye globe, are only induced by variations of the fluid stresses acting on the membrane. Finally, we assume small oscillations about the undisturbed position, so that the displacement of each point of the membrane is aligned with the direction of the $\zeta^*$-axis.

According to the above hypotheses the equation governing the motion of the membrane can be directly written in terms of the cylindrical coordinates $(\rho^*, \vartheta, \zeta^*)$:

$$\frac{\partial^2 \eta^*}{\partial t^{*2}} - c^2 \left( \frac{\partial^2 \eta^*}{\partial \rho^{*2}} + \frac{1}{\rho^*}\frac{\partial \eta^*}{\partial \rho^*} + \frac{1}{\rho^{*2}}\frac{\partial^2 \eta^*}{\partial \vartheta^2} \right) + \frac{p'^*}{\sigma_m} - \frac{p''^*}{\sigma_m} = 0, \tag{2.7}$$

where $c^2 = T/\sigma_m$, with $\sigma_m$ being the mass of the membrane per unit area, and where $p'^*$ and $p''^*$ represent the pressure at the membrane in the upper and lower portions of the container, respectively. It is noted that, membrane oscillations being of small amplitude, it follows that the elastic stress remains constant.

We must explicitly point out at this stage that the above formulation of the membrane equation does not *a priori* ensure that the membrane oscillations satisfy the physical constraint that the fluid volume must be preserved. However, it will be shown below that the solution forced by the sphere rotations, which is of interest for the present work, satisfies this requirement.

We consider small-amplitude periodic oscillations of the container; hence we write

$$\beta(t^*) = \delta \sin(\omega_o t^*), \qquad \delta \ll 1. \tag{2.8$a,b$}$$

Furthermore, in order to formulate the problem in dimensionless form, we define the following dimensionless variables:

$$(\rho, \zeta, \eta) = \frac{(\rho^*, \zeta^*, \eta^*)}{R}, \qquad \phi = \frac{\phi^*}{\omega_c R^2}, \qquad t = \omega_c t^*, \qquad p = \frac{p^*}{\rho_v \omega_c^2 R^2}. \tag{2.9$a$–$d$}$$

In the above expressions $\omega_c$ represents a characteristic scale of the frequency of oscillations of the membrane. In the following both free oscillations and oscillations forced by eye movements will be considered. In the latter case $\omega_c$ has to be chosen equal to the frequency of eye rotations $\omega_o$. In the case of free membrane oscillations the choice of $\omega_c$ has been based on scaling arguments. By balancing the order of magnitude of the first term in equation (2.7) with that of the terms involving spatial derivatives, we find

$$\omega_c = \frac{c}{R}. \tag{2.10}$$

The assumption of small-amplitude rotations of the container, mathematically expressed by (2.8*b*), allows us to linearize the problem. We then introduce the following expansion:

$$(\phi, \eta, p) = \delta(\phi_1, \eta_1, p_1) + O(\delta^2), \tag{2.11}$$

and keep only the order-$\delta$ terms in all the equations. We thus derive the dimensionless linearized form of equations (2.1), (2.2) and (2.3), written in terms of the coordinate system $(\rho, \vartheta, \zeta)$:

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \phi_1}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \phi_1}{\partial \vartheta^2} + \frac{\partial^2 \phi_1}{\partial \zeta^2} = 0, \tag{2.12a}$$

$$\frac{\partial \phi_1}{\partial \rho}\rho + \frac{\partial \phi_1}{\partial \zeta}\zeta = 0 \qquad (\rho^2 + \zeta^2 = 1), \tag{2.12b}$$

$$\frac{\partial \eta_1}{\partial t} - \frac{\partial \phi_1}{\partial \zeta} + \rho \sin\vartheta \cos t = 0 \qquad (z = 0). \tag{2.12c}$$

Finally, Bernoulli's equation (2.4) takes the following linearized form:

$$p_1 + \frac{\partial \phi_1}{\partial t} = 0. \tag{2.13}$$

## 3. Solution for the flow field

We now expand the function $\eta_1(\rho, \vartheta, t)$, representing the membrane displacement, in Fourier–Bessel series. The structure of the forcing term appearing in equation (2.12*c*) suggests that only the first mode in the azimuthal direction is directly forced by eye rotations, therefore only this azimuthal mode will be considered. We then write

$$\eta_1(\rho, \vartheta, t) = \sum_{m=1}^{\infty} e_m(t) \sin\vartheta\, J_1(\alpha_m \rho), \tag{3.1}$$

where $J_1$ represents the order-1 Bessel function of the first kind and $\alpha_m$ $(m = 1, 2, \ldots, \infty)$ represent the zeros of such a function. Notice that the constraint whereby the fluid volume must be preserved is indeed satisfied by the above expansion. Moreover, we note that, since the $\alpha_m$ are the zeros of the Bessel function $J_1$, it follows that $\eta_1$ automatically vanishes at the boundary $(\rho = 1)$.

The boundary condition at the membrane (2.12*c*) suggests expanding the velocity potential in the form

$$\phi_1(\rho, \vartheta, \zeta, t) = \sum_{m=1}^{\infty} \psi_m(\rho, \zeta) \sin\vartheta \frac{\mathrm{d}e_m(t)}{\mathrm{d}t} + \chi(\rho, \zeta) \sin\vartheta \cos t, \tag{3.2}$$

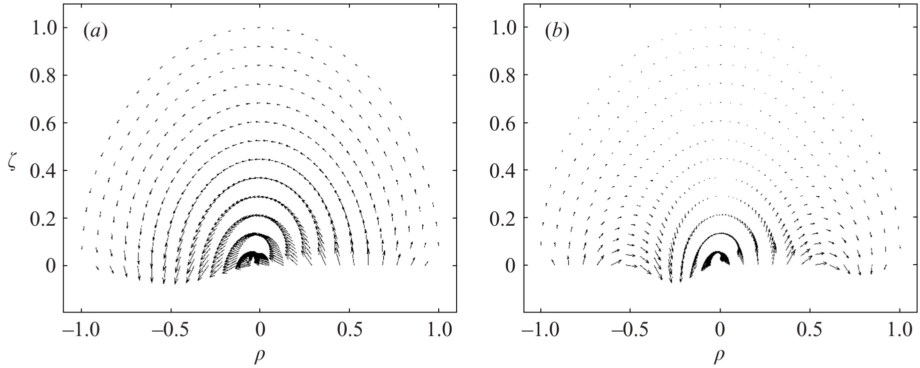


FIGURE 3. Spatial structure of the flow field induced by oscillation of (*a*) mode 1 and (*b*) mode 2 of the membrane in the plane identified by the line $\vartheta = \pi/2, 3/2\pi$ and the $\zeta$-axis. The velocity field is computed as the gradient of the functions $\psi_1$ and $\psi_2$, respectively. Vector length scales in the two plots are the same.

where $\phi_1$ is decomposed into two parts: one forced by the membrane oscillations and the other, proportional to the function $\chi$, forced by the container rotation.

Substituting from the expansions (3.1) and (3.2) into the differential problem (2.12*a*–*c*), we find the following differential problems for $\psi_m$ $(m = 1, 2, \ldots, \infty)$ and $\chi$:

$$\frac{\partial^2 \psi_m}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \psi_m}{\partial \rho} + \frac{\partial^2 \psi_m}{\partial \zeta^2} - \frac{\psi_m}{\rho^2} = 0, \tag{3.3a}$$

$$\rho \frac{\partial \psi_m}{\partial \rho} + \zeta \frac{\partial \psi_m}{\partial \zeta} = 0 \qquad (\rho^2 + \zeta^2 = 1), \tag{3.3b}$$

$$\frac{\partial \psi_m}{\partial \zeta} = J_1(\alpha_m \rho) \qquad (\zeta = 0), \tag{3.3c}$$

$$\psi_m = 0 \qquad (\rho = 0). \tag{3.3d}$$

$$\frac{\partial^2 \chi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \chi}{\partial \rho} + \frac{\partial^2 \chi}{\partial \zeta^2} - \frac{\chi}{\rho^2} = 0, \tag{3.4a}$$

$$\rho \frac{\partial \chi}{\partial \rho} + \zeta \frac{\partial \chi}{\partial \zeta} = 0 \qquad (\rho^2 + \zeta^2 = 1), \tag{3.4b}$$

$$\frac{\partial \chi}{\partial \zeta} = \rho \qquad (\zeta = 0), \tag{3.4c}$$

$$\chi = 0 \qquad (\rho = 0). \tag{3.4d}$$

The above problems are solved numerically through a second-order finite difference scheme using a system of polar coordinates in the $(\rho, \zeta)$-plane. The solution procedure is based on the ADI method which leads to solving a sequence of tridiagonal algebraic systems obtained by sweeping the computational mesh by rows and then by columns (see for instance Tannehill, Anderson & Pletcher 1997). A mesh of $80 \times 80$ points has been adopted; numerical tests indicate that, with a computational mesh of $160 \times 160$ points, results do not change up to the fourth significant figure. Moreover, we have assumed that convergence was reached when the relative error on the solution was lower that $10^{-8}$.

In figure 3(*a,b*) the spatial structure of flow fields obtained by the numerical solution of the Laplace equation for $\psi_m$ is shown in the plane identified by the line

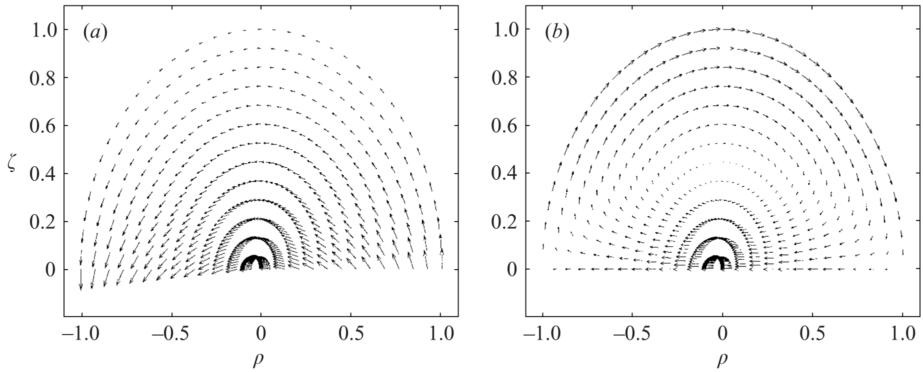


FIGURE 4. Spatial structure of the flow field induced by container rotations in the plane identified by the line $\vartheta = \pi/2, 3/2\pi$ and the $\zeta$-axis; the membrane is kept rigid. The absolute velocity field is computed as the gradient of the function $\chi$: (*a*) absolute velocity, (*b*) relative velocity. Vector length scales in the two plots are the same.

$\vartheta = \pi/2, 3/2\pi$ and the $\zeta$-axis for the first two spatial modes of oscillation of the membrane. It is noted that, as the spatial mode increases, the effect of membrane oscillations on the flow field diminishes increasingly quickly far from the membrane. In other words only oscillations of the first few modes give rise to significant velocities within the vitreous body.

The flow field induced by container rotations with the membrane kept rigid is obtained by solving the problem for the unknown function $\chi$. In figure 4(*a, b*), the spatial structure of the absolute and relative velocity fields is shown in the plane identified by the line $\vartheta = \pi/2, 3/2\pi$ and the $\zeta$-axis. Figure 4(*b*) shows that the relative velocity generates a circulation cell, with a sense of rotation opposite to that of the container.

## 4. Solution of the equation of the membrane

### 4.1. *Free oscillations*

Let us next consider the problem of determining the free oscillations of the membrane. To this end we assume the container wall to be still, which mathematically implies that the function $\chi$ vanishes identically.

As pointed out above, the characteristic frequency of free membrane oscillations $\omega_c$ can be estimated by equation (2.10). With such a choice equation (2.7) takes the following dimensionless form:

$$\frac{\partial^2 \eta_1}{\partial t^2} - \left( \frac{\partial^2 \eta_1}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \eta_1}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial \eta_1}{\partial \vartheta^2} \right) + \Gamma_1 (p_1' - p_1'') = 0, \tag{4.1}$$

where $\Gamma_1$ is the only controlling dimensionless parameter of the problem, defined as

$$\Gamma_1 = \frac{\rho_v R}{\sigma_m}. \tag{4.2}$$

In order to solve equation (4.1) we need to compute the pressure distribution at the membrane, i.e. at $\zeta = 0$, using the equation (2.13). It is noted that the structure of expansion (3.1) suggests that the flow field generated by the membrane oscillations, and consequently the pressure distribution acting on the membrane, are antisymmetric about the membrane. Therefore, the total stress $p_1' - p_1''$ acting on the membrane can

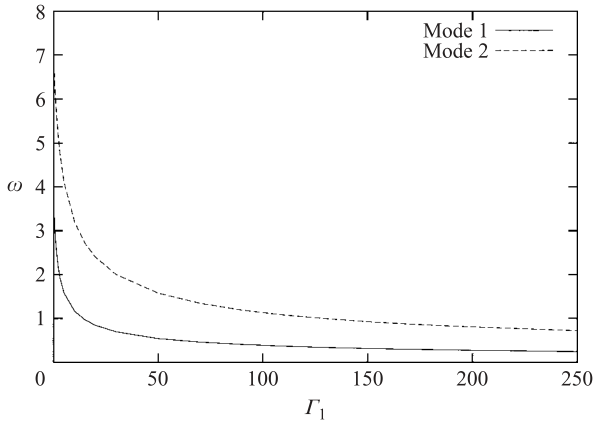


FIGURE 5. The frequencies of oscillation of modes 1 and 2 of the membrane plotted versus the dimensionless parameter $\Gamma_1$.

be written as $2p_1$ where we denote by $p_1$ the pressure of the upper fluid, evaluated at the membrane.

To solve equation (4.1) it is convenient to expand in Bessel series the potential functions $\psi_m$, evaluated at the membrane, in the form

$$\psi_m(\rho, 0) = \sum_{k=1}^{\infty} \hat{\psi}_{mk} J_1(\alpha_k \rho). \tag{4.3}$$

According to equations (2.13), (3.2) and (4.3) the pressure distribution acting at the membrane can be written as

$$p_1 = -\sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \hat{\psi}_{mk} J_1(\alpha_k \rho) \sin\vartheta \frac{\mathrm{d}^2 e_m(t)}{\mathrm{d}t^2}. \tag{4.4}$$

Substituting (3.1) and (4.4) into (4.1) we find, after some manipulations involving properties of the Bessel functions, the following system of homogeneous linear ordinary differential equations:

$$\ddot{e}_m + \alpha_m^2 e_m - 2\Gamma_1 \sum_{k=1}^{\infty} \hat{\psi}_{km} \ddot{e}_k = 0, \tag{4.5}$$

where an over dot denotes time derivatives. The above system is best written in matrix form as

$$\boldsymbol{M}\ddot{\mathbf{e}} + \boldsymbol{K}\mathbf{e} = 0, \tag{4.6}$$

where $\boldsymbol{M}$ is referred to as the 'mass matrix' and $\boldsymbol{K}$ as the 'stiffness matrix'. Due to the non-dissipative nature of the problem the dynamical behaviour of the system consists of pure oscillations. The frequencies of oscillation are given by the square roots of the eigenvalues of the matrix $\boldsymbol{M}^{-1}\boldsymbol{K}$.

In figure 5 the dimensionless natural frequencies of oscillation $\omega$ of the first two modes of the membrane are plotted versus the parameter $\Gamma_1$. Notice that, for vanishing values of $\Gamma_1$, we recover the case of free membrane oscillations in the absence of fluid. In such a case it is well known that the natural frequencies of oscillation are given by the zeros of the Bessel function. For increasing values of the parameter $\Gamma_1$ the dimensionless frequencies of oscillation significantly decrease. In other words, as could be expected by recalling results obtained in different contexts (see for instance Gottlieb & Aebisher 1986, 1989), when the membrane vibrates in the fluid the characteristic frequencies decrease. Such an effect indicates a strong coupling between the fluid and the membrane dynamics, even in a linear context.

The thickness of vitreous membranes typically ranges between tens and hundreds of microns. With the membrane density about $1000\,\mathrm{Kg\,m^{-3}}$, the dimensionless parameter $\Gamma_1$ takes values ranging from 50 to 200. It is noted that, within this range of values of $\Gamma_1$, the dimensionless frequencies of oscillations do not display a strong variability. Obviously, as far as the dimensional frequencies are concerned, the above results crucially depend on the intensity of the stretching force per unit length $T$ acting on the membrane: the larger $T$ is, the higher the natural frequencies of oscillation. Unfortunately, to our knowledge, no measurements of $T$ are available yet; indeed direct *in vivo* measurements of the stretching force acting on a vitreous membrane are almost impossible. However, we recall that, according to the medical literature, the stress acting on the membrane is often large enough to cause a retinal detachment. The present work concerns membranes which have not yet caused a retinal detachment and it is therefore possible to estimate an upper bound for the stress acting on the membrane based on measurements of the adhesive strength between the retina and the epithelium. According to Wu, Peters & Hammer (1987) the failure load per unit length of retina ranges between 0.4 and $0.6\,\mathrm{N\,m^{-1}}$; moreover the authors state that "...the tensile strength of retina is roughly twice the adhesive strength of retina to pigment epithelium". Therefore, one may expect that the maximum stretching force per unit length $T$, which does not cause a retinal detachment, would be in the range of about 0.2–0.3 $\mathrm{N\,m^{-1}}$. Using for $T$ the value $0.1\,\mathrm{N\,m^{-1}}$, the dimensional frequency of oscillation of the first mode takes values of about tens of Hz, i.e. values comparable with the frequency of microsaccadic eye movements, which range between 15 and 30 Hz.

### 4.2. *Forced oscillations*

We now consider the motion of the membrane forced by eye rotations. In this case, the characteristic frequency of oscillations being equal to $\omega_o$, equation (2.7) takes the following dimensionless form:

$$\frac{\partial^2 \eta_1}{\partial t^2} - \frac{1}{\Gamma_2^2}\left(\frac{\partial^2 \eta_1}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \eta_1}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial \eta_1}{\partial \vartheta^2}\right) + 2\Gamma_1 p_1 = 0, \tag{4.7}$$

where $\Gamma_2$ is the dimensionless parameter

$$\Gamma_2 = \frac{\omega_o R}{c}, \tag{4.8}$$

representing the dimensionless forcing frequency. A particular solution of equation (4.7) is obtained from equation (3.1) by setting

$$e_m(t) = g_m \sin t, \tag{4.9}$$

where $g_m$ are constants to be determined.

In analogy with the approach employed for the functions $\psi_m$, the potential function $\chi$ computed at the membrane, is also expanded in Bessel series as

$$\chi(\rho, 0) = \sum_{m=1}^{\infty} \hat{\chi}_m J_1(\alpha_m \rho), \tag{4.10}$$

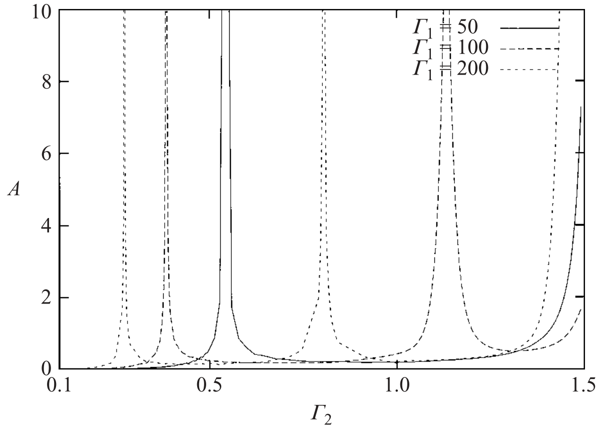


FIGURE 6. The amplitude $A$ of membrane oscillations is plotted versus the dimensionless parameter $\Gamma_2$ for different values of $\Gamma_1$.

and the pressure distribution at the membrane can be written in the form

$$p_1 = \sum_{m=1}^{\infty} J_1(\alpha_m \rho) \left( \hat{\chi}_m + \sum_{k=1}^{\infty} \hat{\psi}_{km} g_k \right) \sin t \sin \vartheta. \tag{4.11}$$

Substituting (3.1), (4.9) and (4.11) into (4.7) we obtain the following non-homogeneous linear algebraic system:

$$g_m \left(1 - \frac{\alpha_m^2}{\Gamma_2^2}\right) - 2\Gamma_1 \sum_{k=1}^{\infty} \hat{\psi}_{km} g_k = 2\Gamma_1 \hat{\chi}_m, \tag{4.12}$$

whose solution is readily found numerically.

In figure 6 the amplitude $A$ of membrane oscillations, calculated by taking into account the first 20 modes, is plotted versus $\Gamma_2$ for given values of $\Gamma_1$. Realistic values of the parameter $\Gamma_2$, corresponding to microsaccadic frequencies and to values of the force per unit length $T$ of order $0.1\,\mathrm{N\,m^{-1}}$ or less, are expected to be larger than 0.1. Moreover, since for the present analysis to be valid we need $T$ to be sufficiently large, computations have been restricted to the range $0.1 < \Gamma_1 < 1.5$.

Figure 6 shows that periodic rotations of the container may resonantly excite membrane oscillations. Resonance occurs when the forcing frequency is close to one of the natural frequencies of the system. Close to resonant conditions, the present linear model fails and a nonlinear approach is required to evaluate the actual amplitudes experienced by the membrane oscillations. If we start from a very low forcing frequency and then progressively increase it, we first find the resonant excitation of the first spatial mode. Further increasing $\Gamma_2$, resonance of all higher modes is progressively excited, as clearly shown in figure 6.

On the basis of the considerations reported above regarding the order of magnitude of the physical quantities involved in the problem, we can finally argue that microsaccadic eye movements may resonantly excite membrane oscillations for values of the membrane stress that can easily be encountered in practice.

Finally, in figure 7($a,b$), typical spatial structures of the absolute flow fields are shown in the plane identified by the line $\vartheta = \pi/2, 3/2\pi$ and the $\zeta$-axis for two different sets of values of the dimensionless parameters $\Gamma_1$ and $\Gamma_2$, corresponding to sub-resonance ($a$) and super-resonant ($b$) conditions. The differences between the two

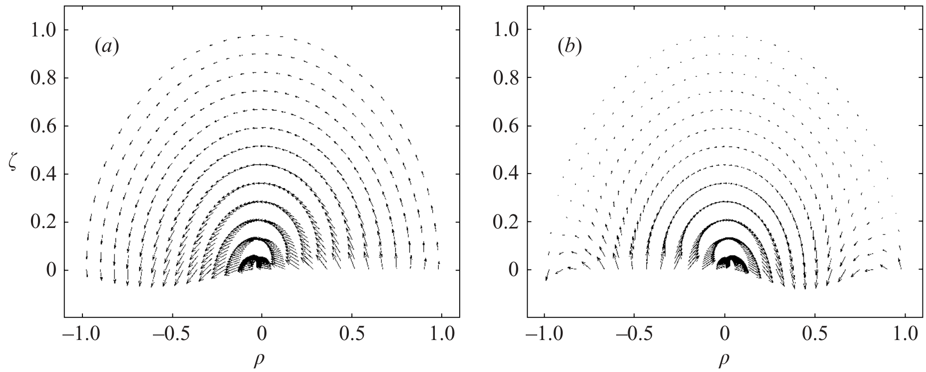


FIGURE 7. Spatial structure of the flow field induced by container rotations in the plane identified by the line $\vartheta = \pi/2, 3/2\pi$ and the $\zeta$-axis: (*a*) $\Gamma_1 = 100$, $\Gamma_2 = 0.32$; (*b*) $\Gamma_1 = 100$, $\Gamma_2 = 0.58$.

flow fields, which are best appreciated close to the membrane, are due to the different phase of membrane oscillations relative to the eye movements in the two cases. In other words we recover the well-known feature of linear oscillators (see for instance Thompson & Stewart 1986) that they display a sudden phase shift of the oscillator response relative to the forcing when the resonance frequency is crossed.

## 5. Final remarks

We have formulated a somewhat idealized model of the dynamics of vitreous membranes driven by microsaccadic eye movements. The main ingredients of the model are as follows. The vitreous has been treated as an incompressible fluid in irrotational motion and the membrane has been assumed to be impermeable, elastic and pre-stressed.

In spite of its simplicity, the present model provides some insight into the oscillatory behaviour of the membrane–vitreous system. In particular, the analysis shows that, as expected, the presence of the fluid strongly decreases the natural frequencies of oscillation of the membrane. The model also shows that microsaccadic eye movements may resonantly excite membrane oscillations for values of the membrane stress $T$ which may easily occur in practice. Under resonant conditions the amplitude of membrane oscillations attains large values, hence large dynamic stresses are expected to occur in the membrane, possibly causing retinal detachment. The above results suggest extending the present investigation in order to ascertain the actual clinical relevance of such findings.

In particular, in order to formulate a more realistic model, it will be necessary to remove some of the constraints imposed in the present work:

the amplitude of the membrane oscillations should be allowed to attain finite values, calling for a fully nonlinear treatment of the problem;

the effect of removing the pre-stressed character of the membrane should be analysed;

the possibility that the rheological behaviour of vitreous on one or both sides of the membrane be viscoelastic will also require treatment;

finally, it may be worth investigating the possible effect of membrane permeability on the vitreous dynamics.

Some of the above developments would benefit from an experimental investigation performed on a large-scale model of the eye.

Rodolfo Repetto thanks Professor A. Luongo for many helpful suggestions during the development of the analysis and acknowledges useful discussions with Dr L. Carassale, Professor M. Vasta and Dr F. Grillo. Particular thanks is also due to Professor G. Ciurlo for drawing our attention on the clinical relevance of the problem considered in the present paper. The constructive criticism of anonymous referees has helped improving the presentation of the paper. The work has been funded by MIUR, under the project FIRB (RBAU01Z44F_005).

## REFERENCES

Ashe, J., Hain, T., Zee, D. & Schatz, N. 1991 Microsaccadic flutter. *Brain* **114**, 461–472.

Courant, R. & Hilbert, D. 1937 *Methods of Mathematical Physics*, vol. 1. Oxford University Press.

David, T., Smye, S., Dabbs, T. & James, T. 1998 A model for the fluid motion of vitreous humour of the human eye during saccadic movement. *Phys. Med. Biol.* **43**, 1385–1399.

David, T., Smye, S., Dabbs, T., James, T. & Dabbs, T. 1997 Time-dependent stress and displacement of the eye wall tissue of the human eye. *Med. Engng Phys.* **19**, 131–139.

González, G. & Fitt, A. 2003 The mathematical modelling of human eye – a PHD study. *Mathematics Today*, February, 4–9.

Gottlieb, H. P. W. & Aebisher, H. A. A. 1986 Eigenfrequencies shifts of a baffled circular membrane in a fluid medium. *Acustica* **61**, 223–232.

Gottlieb, H. P. W. & Aebisher, H. A. A. 1989 Vibration frequencies of a baffled annular membrane in a fluid medium. *Acustica* **69**, 241–248.

Lee, B., Litt, M. & Buchsbaum, G. 1992 Rheology of the vitreous body. Part I: viscoelasticity of human vitreous. *Biorheology* **29**, 521–533.

Klintworth, G. K. & Scroggs, M. W. 1997 Normal eye and ocular adnexa. In *Histology for Pathologists*, 2nd Edn (ed. S. Sternberg), Chapter 13. Lippincott-Raven, Philadelphia.

Tannehill, J. C., Anderson, D. A. & Pletcher, R. H. 1997 *Computational Fluid Mechanics and Heat Transfer*, 2nd Edn. Taylor & Francis.

Thompson, J. M. T. & Stewart, H. B. 1986 *Nonlinear Dynamics and Chaos*. John Wiley and Sons.

Tolentino, F. I., Schepens, C. L. & Freeman, H. M. 1976 *Vitreoretinal Disorders – Diagnosis and Management*. Saunders W. B. Company.

Wu, W., Peters, W. H. & Hammer, M. E. 1987 Basic mechanical properties of retina in simple elongation. *J Biomech Engng* **109** (1), 65–67.